Analytical solitons in nonlinear transmission lines loaded with heterostructure barrier varactors

X. Oriols and F. Martín
Departament d’Enginyeria Electrònica, Universitat Autònoma de Barcelona, 08193-Bellaterra, Spain

(Received 2 January 2001; accepted for publication 4 June 2001)

The propagation of solitons in transmission lines periodically loaded with nonlinear capacitors is analyzed. It is demonstrated that the dependence of soliton amplitude on propagating velocity can be computed with a simple equation, valid for any arbitrary nonlinear capacitance. By means of an approximation to the nonlinear reactance, an analytical expression that completely describes soliton profiles is presented. This can be applied to obtain moderate to high-amplitude soliton wave forms in nonlinear transmission lines (NLTLs) loaded with heterostructure barrier varactors, where standard approaches do not hold. This model can be of help in understanding harmonic generation in the terahertz range using monolithic NLTLs as frequency multipliers. © 2001 American Institute of Physics. [DOI: 10.1063/1.1388863]

I. INTRODUCTION

Recently, the terahertz region of the electromagnetic spectrum (loosely defined as the range between 100 GHz and 10 THz) has become very attractive for future wide bandwidth systems. In this regard, electronic engineers are now exploring different possibilities to develop signal sources operating at these frequencies. Active THz solid state oscillators (such as those based on resonant tunneling diodes) only provide small amounts of power above 100 GHz. In parallel, the interest in monolithic nonlinear transmission lines (NLTLs) has grown due to their ability to give acceptable power efficiencies at THz frequencies by harmonic multiplication. Essentially, a NLTL is a ladder network consisting of a high-impedance propagating medium periodically loaded with nonlinear capacitive devices, such as Schottky diodes or heterostructure barrier varactors (HBVs). The multiplicative process in NLTLs is understood as a direct consequence of soliton-like propagation in this medium. Qualitatively, the origin of solitons in NLTLs is explained by the balance between the effect of dispersion (due to the periodic location of capacitors in the NLTL) and nonlinearity (due to the voltage dependence of the capacitances). A soliton is a localized wave form that travels along the system with constant velocity and undeformed shape (see Ref. 7 for a complete review on this topic). It is well known that in transmission media supporting solitons, any input pulse with a duration greater than soliton width tends to dissolve into a superposition of solitons. In this regard, a sinusoidal signal fed to the NLTL will progressively decompose into multiple solitons per cycle, and harmonics of the input frequency will be obtained at the output. Since monolithic NLTLs support solitary waves of picosecond duration, output signals with high frequency harmonic content (hundreds of GHz and THz) can be produced with NLTL-based multipliers.

In this work we will present a method to study solitary waves in NLTLs. We will focus our attention on NLTLs loaded with HBVs that have been recently proposed “as the best overall solution for third harmonic generation,” since the symmetric capacitance–voltage ($C–V$) characteristic that HBVs exhibit directly suppresses even order harmonics. We will show that our model can be applied to obtain moderate and high pulse wave forms (large signal conditions), where prior approaches do not hold. This article is divided as follows: in Sec. II previous works on soliton propagation in NLTLs are summarized. In Sec. III a very simple equation describing soliton amplitude versus propagation speed, valid for any experimental $C–V$ curve of the nonlinear device, is derived. In Sec. IV, by means of a two-piece linear approximation for the $C–V$ curve of the HBV, an analytical expression that completely describes soliton wave forms will be presented. Section V will be devoted to test the range of validity of our proposal by comparing it with numerical results. Finally, the main conclusions will be presented in Sec. VI.

II. PREVIOUS WORK ON SOLITON PROPAGATION IN NLTLs

A NLTL is a network consisting of nonlinear capacitors periodically placed between transmission line sections [see Fig. 1(a)]. A simple model that retains most of the physics of the NLTL is the $L C$ equivalent circuit of Fig. 1(b). An inductor $L$ and a constant capacitance $C_o$ (Ref. 9) model each section of the transmission line. This model is accurate enough as long as the minimum wavelength of the propagating signal is longer than the distance between nonlinear capacitances. Recently, a more sophisticated model, using the microwave circuit theory to explicitly simulate each transmission line section, has been developed to study high frequency components of picosecond pulse generation with NLTLs. However, to study soliton-based frequency multiplication at millimeter wavelengths and to analyze the effects of the main parameters, the lumped element equivalent circuit of NLTLs is adequate and has been commonly used.

---

*Electronic mail: xavier.oriols@uab.es*
The theoretical study of soliton propagation in nonlinear LC networks has been carried out during the last three decades. Much of the attention has been devoted to two equations: the Toda lattice equation\(^{11}\) and the Korteweg–de Vries (KdV) equation.\(^{12}\) In 1973, Hirota and Suzuki\(^ {13}\) demonstrated that the LC ladder network with a nonlinear shunt capacitance given by the expression: 

\[ C(V) = C_o (V - V_o) \]

(\(C_o\) is a constant) is equivalent to the Toda lattice, for which soliton solutions are well known.\(^{11}\) They even reported experimental results for the transmission of solitons in LC networks by using varactor diodes with nonlinearity given by 

\[ C(V) \approx 27(V - V_o) - 0.48 \text{pF} \]

Recently, Singer and Oppenheim\(^ {14}\) have proposed a new hardware implementation of the Hirota and Suzuki capacitor, providing a more accurate experimental implementation of the Toda lattice with electrical circuits. Although much of the work related to nonlinear lumped LC networks has been done with the Toda lattice (and its particular \(C - V\) characteristic) a discrete KdV equation, sometimes referred to as the nonlinear ladder equation, has been also studied and experimentally implemented.\(^ {14}\)

However, rather than finding a capacitor that transforms a \(LC\) network to a particular well-known nonlinear equation (as has been done by Hirota and Suzuki), the main interest is to know which are the soliton solutions that correspond to \(LC\) systems with experimental \(C - V\) curves. In this regard, the general and traditional analysis of solitons in \(LC\) networks has been related to the KdV equation. Under this equation, the evolution of solitary waves can be exactly known through a linear calculation know as “inverse method.”\(^ {17}\) A continuum limit of the actual lattice equation and a weakly nonlinearity are implicitly considered when the KdV equation is used to describe \(LC\) networks. In this framework, several nonlinearities such as 

\[ -\varepsilon \cdot V \] or \[ \pm \varepsilon \cdot V^2 \]

have been considered to design monolithic\(^ {12}\) NLTLS based on Schottky-barrier or metal–insulator–semiconductor structures. By exploring similarities with KdV solutions, analytical expressions for soliton wave form description are provided (for a complete review see Ref. 15).

In accordance to fabricated NLTLS prototypes, previous theoretical works are mainly based on Schottky diodes as the nonlinear device. Since these diodes need to be reverse-biased, the use of small signal conditions around a dc value is fairly justified. Recently, HBVs have been proposed as the nonlinear candidates for harmonic generation in NLTLS. HBVs have a key advantage: only odd order harmonics of the input frequency are produced due to the symmetric \(C - V\) characteristic of the HBV.\(^ {16,17}\) This is very important from the point of view of conversion efficiency optimization and simplifies the fabrication of higher order multiplier circuits. Excellent HBV tripler results demonstrating an efficiency of 12% at 247 GHz have been reported by Melique et al.\(^ {18}\)

In the present work, by means of the \(LC\) network model, we will study soliton propagation characteristics in NLTLS loaded with HBVs. In particular, we will provide analytical expressions for the description of moderate and high amplitude solitons. To our knowledge, this particular topic has not been dealt with in the literature. In the next section we will present a simple relationship, valid for any nonlinear capacitor, for the soliton amplitude dependence on propagating velocity.

### III. A SIMPLE MODEL TO CHARACTERIZE NRTL SOLITONS

The elementary \(LC\) network depicted in Fig. 1(b) is described by the following equation:

\[
C(V_i(t)) \frac{d^2 V_i(t)}{dt^2} + \left( \frac{dV_i(t)}{dt} \right)^2 \frac{dC(V)}{dV} = \frac{V_{i+1}(t) - 2 \cdot V_i(t) + V_{i-1}(t)}{L},
\]

(1)

where \(C(V) = C_o + C_p(V)\) is the parallel combination of the per-section and nonlinear capacitance. In order to simplify this equation, we employ a standard continuum limit\(^ {19}\) for the voltage, i.e., \(V_i(t) \rightarrow V(x,t)\), where \(x = i \cdot d\), and \(d\) is the distance between nonlinear capacitors. Within this limit, a Taylor expansion up to fourth order can be carried out:

\[
V_i(t) = V(x \pm d,t) \approx V(x,t) \pm \frac{\partial V}{\partial x} \cdot d + \frac{1}{2} \left( \frac{\partial^2 V}{\partial x^2} \right) \cdot d^2 \pm \frac{1}{3!} \left( \frac{\partial^3 V}{\partial x^3} \right) \cdot d^3 + \frac{1}{4!} \left( \frac{\partial^4 V}{\partial x^4} \right) \cdot d^4.
\]

(2)

Moreover, solitons moving at a propagating velocity, \(v\), have a static profile in a new time coordinate system given by \(s = t - x/v\). With this new variable \(\partial \partial s = -1/v \cdot \partial \partial t \cdot \partial \partial s\) and \(\partial \partial t = d/dt\). In this way, Eq. (1) can be written as

\[
L \frac{d}{ds} \left( \frac{C(V)}{dV(s)/ds} \right) = T^2 \frac{d^2 V(s)}{ds^2} + \frac{T^4}{12} \frac{d^4 V(s)}{ds^4}
\]

(3)

where \(T\) is the per-section propagation delay, related to soliton velocity by \(v = d/T\). If Eq. (3) is twice integrated, we obtain:

\[
F[V(s)] = \frac{T^2}{L} \cdot V(s) = \frac{T^4}{12L} \frac{d^2 V(s)}{ds^2},
\]

(4)
where the integration constants are set to zero because we are looking for localized solutions, whose derivatives tend to zero when \( s \to \pm \infty \). \( F(V) \) is a function responsible for non-linearity defined as

\[
F(V) = \int_0^V C(V')dV'.
\]

(5)

In the literature, only specific \( C-V \) characteristics of nonlinear devices have been considered.\(^{15,19}\) We have recently proposed a very simple algorithm\(^{20}\) to obtain soliton solutions under arbitrary \( C-V \) characteristics by numerically solving Eq. (4). Nevertheless, in order to obtain the dependence of soliton amplitude on \( T \), it is not necessary to solve the differential equation. To show this, Eq. (4) is multiplied by \( dV(s)/ds \) and integrated in \( s \):

\[
\int_0^\infty \left[ F[V(s)] - \frac{T^2}{L} V(s) \right] \frac{dV(s)}{ds} ds = 0,
\]

(6)

The right-hand side of Eq. (6) can be integrated by parts and found to be equal to zero for any function satisfying \( dV(s)/ds = 0 \) at \( s=0 \) and \( s=\infty \) (these are the boundary conditions for soliton solutions). By transforming the left-hand side of Eq. (6) to a voltage integral we finally obtain:

\[
\int_0^{V_{\text{max}}} \left[ F(V) - \frac{T^2}{L} V \right] dV = 0,
\]

(7)

where it is assumed that \( V(s=\infty) = 0 \), and \( V(s=0) = V_{\text{max}} \) is defined as the soliton amplitude. Integrating Eq. (7), \( T \) can be isolated and compactly expressed as

\[
T = \sqrt{2 \cdot L \cdot G(V_{\text{max}})} / V_{\text{max}},
\]

(8)

where \( G(V) \) is related to the previously defined function \( F(V) \) by

\[
G(V) = \int_0^V F(V')dV'.
\]

(9)

By means of Eq. (8), the dependence of per-section propagation delay on soliton amplitude, or vice versa, can be easily obtained without the need to solve any differential equation. This relationship allows one to analyze the sensitivity of propagation velocity (or \( T \)) to soliton amplitude as a function of NLTL parameters. As has been recently pointed out by the authors,\(^{20}\) this is very important for the optimization of NLTL-based frequency multipliers, since a strong sensitivity of \( T \) to soliton amplitude means that the decomposition of the feeding signal into solitons proceeds faster. This improves output power, since an important reduction of device and transmission line losses is obtained by shortening NLTL length.

**IV. ANALYTICAL EXPRESSION FOR SOLITONS**

In Fig. 2 we have represented the measured \( C-V \) characteristic of a dual heterostructure diode developed as a prototype for harmonic generation at IEMN-Lille.\(^{21}\) An analytical approximation for the experimental \( C-V \) curve is represented by a dashed line. Within this approximation, the total capacitance \( C(V) \), defined as the parallel combination of per-section \( C_o \) (Ref. 9), and device capacitance \( C_D(V) \), can be written as

\[
C(V) = C_o + C_d \cdot \text{sech}(k \cdot V),
\]

(10)

where \( C_d \) is defined in the figure and \( k \) determines the \( C(V) \) width. Following the analysis of the previous section, in Fig. 3 we have depicted (with a dashed line) the function \( F(V) \) obtained from the \( C(V) \) described above. In order to obtain analytical soliton solutions from the model presented in the previous section, we proceed by fitting \( F(V) \) with a two-piece linear approximation (see the solid line in Fig. 3). First of all, let us notice that there is not a general criterion to set the slope and the origin of the two straight lines [i.e., to minimize the error between \( F(V) \) and its approximation]. If we were interested in small-voltage solitons it would be necessary to accurately approximate \( F(V) \) at low voltages. However, as the main concern of our work is the description of moderate to high voltage solitons, an asymptotic linear approximation as shown in Fig. 3 is appropriate. Following this criterion, we approximate the \( F(V) \) by two straight lines:

![Fig. 2. Measured C–V curve for a dual heterostructure diode (Ref. 20) (solid line) and analytical approximation described by Eq. (10) (dashed line).](image)

![Fig. 3. Representation of F(V) obtained from an analytical capacitance vs voltage curve (dashed line) and the two-piece linear approximation of F(V) presented in this work (solid line). The parameters of the analytical C–V characteristic are given by C_o=43.5 fF, C_d=120 fF, and k=0.5. The dotted trace corresponds to a line with slope T^2/L, where the per-section propagation delay is T=2.75 ps and L=108.7 pH.](image)
where the soliton amplitude is defined as

$$V = \frac{1}{C_M} V, \quad 0 < V < V_1$$

$$C_o \cdot V + b, \quad V \geq V_1.$$  \hfill (11)

The negative part of $F(V)$ can be inferred by considering the symmetric behavior of $C(V)$, so that $F(-V) = -F(V)$. The parameter $C_M$ is defined as the maximum value of the total capacitance, i.e., $C_M = C_o + C_d$. The voltage $V_1$ is the value at the intersection between the two solid lines (from the figure we obtain $V_1 = b(C_d)$). The parameter $b$ is determined as $b = C_o A / C_M$, where $A$ is defined as

$$A = C_o \cdot V_1 + \int_0^\infty C_D(V) \cdot dV.$$

(12)

Within this two-piece approximation, the nonlinear equation (4) splits into two ordinary second order differential equations whose solutions are quite simple:

$$V(t) = \begin{cases} (V_{\text{max}} - V_2) \cdot \cos(\alpha_1 t) + V_2, & |t| < t_1 \\ \frac{V_1}{e^{-\alpha_2 |t|}} \cdot \exp(-\alpha_2 |t|), & |t| > t_1 \end{cases},$$  \hfill (13)

where $V_{\text{max}}$ is defined by Eq. (8) and $V_2$ is the voltage at the intersection between the second term on the left-hand side of Eq. (4) (dotted line in Fig. 3) and the asymptotic approximation to $F(V)$, i.e., $V_2 = b/(T^2 - LC_o)$. For commodity, the variable $x$ is used instead of $s$ by assuming $x = 0$. On the other hand, the parameters $\alpha_1$ and $\alpha_2$ are defined as $\alpha_1 = (2/T^2) \sqrt{3(T^2 - LC_o)}$ and $\alpha_2 = (2/T^2) \sqrt{3(LC_M - T^2)}$. From these definitions we realize that only solitons propagating with $T$ contained in the interval $\sqrt{LC_o} \leq T \leq \sqrt{LC_M}$ are allowed. Finally, the parameter $t_1$ is defined by $V_1 = V(t_1)$:

$$t_1 = \frac{1}{\alpha_1} \cos^{-1} \left( \frac{V_1 - V_2}{V_{\text{max}} - V_2} \right).$$  \hfill (14)

The time $t_1$ will be always positive since the ratio of the voltages in Eq. (14) must be negative but its modulus smaller than 1.

At this point, although this approach is general and useful for any experimental $C-V$ curve, in order to obtain a compact and simple expression, we will use the analytical expression (10). With this expression we find $A = \pi C_M / 2k$, $V_1 = \pi / 2k$, $V_2 = \pi C_d L / [2k(T^2 - LC_o)]$, and the soliton amplitude is defined as

$$V_{\text{max}} = V_2 \cdot \left(1 + \frac{\sqrt{LC_M - T^2}}{LC_d} \right).$$  \hfill (15)

V. COMPARISON WITH NUMERICAL RESULTS

In order to test the accuracy of the analytical solutions, we have compared them with soliton wave forms obtained by direct numerical simulation of the $LC$ ladder network. We have introduced the analytical solitons as the input signal into a FDTD numerical simulator of a $LC$ ladder network composed of $N = 100$ cells. In Fig. 4 the time evolution of one of these solitons is represented (since the present results are only a numerical test of our model, we have used expression (10) to describe the nonlinearity, rather than an experimental one). We have let solitons evolve and have measured their amplitude after 180 ps. After evolving along 60 cells (i.e., 180 ps) we can consider that the wave form maintains its profile and therefore corresponds to an actual soliton of the network. In Fig. 5 we have compared analytical soliton amplitudes obtained by expression (15) (solid line) to those numerically obtained by means of the method explained before (dots). For high-amplitude solitons (let us remember that this is the motivation of our work) the agreement is almost perfect, while for low voltages the model is less accurate. By inspection of Figs. 3 and 5 we can easily understand this behavior. Small amplitude solitons travel with high per-section transmission times, which means a high slope for the dotted line of Fig. 3. Under these conditions, $V_2$ and $V_1$ are quite near (see Fig. 3) and the difference between $F(V)$ and our two-piece linear approximation is more significative [small amplitude solitons will be better described by two straight lines that approximate $F(V)$ as much as possible for the low voltage range]. Nevertheless, we conclude that for moderate to high amplitude solitons, the agreement between our approach and numerical solutions is excellent.

Finally, taking advantage of Eq. (8), we can compare our model with previous studies without the need to solve any

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{Dependence of soliton amplitude on per-section propagation delay computed using our approach to describe the $C-V$ characteristic. The solid line is obtained within our analytical model and dots are computed by numerically simulating soliton propagation in the $LC$ network. These results are compared with other device nonlinearities depicted in the figure with $\epsilon = 0.05 \text{ V}^{-1}$ (dotted line) and $\epsilon = 0.006 \text{ V}^{-1}$ (dashed line).}
\end{figure}
differential equation. As we have already mentioned in Sec. II, several works considered C–V curves with a nonlinearity proportional to the voltage. Under this assumption it is possible to find analytical soliton solutions:

\[
V(t) = \frac{3}{2 \varepsilon} \frac{v^2 - v_M^2}{v^2} \text{sech} \left( \frac{\sqrt{3}(v^2 - v_M^2)}{v_M} T \right),
\]

where \(v_M^2 = d^2/(LC_M)\). As we have stressed in the Introduction, this approximation is useful for the description of small-voltage variations around a dc value (i.e., small signal conditions), but it does not work in studying high-amplitude solitons propagating in HBV-loaded NLTLs. In order to explain this, Fig. 5 includes the dependence of the soliton amplitude on \(T\) for the following two C–V characteristics. The one already mentioned,

\[C(V) = C_M(1 - 2 \varepsilon \cdot V),\]

and

\[C(V) = C_M(1 - 2 \varepsilon \cdot V^2),\]

Both additional \(V_{\text{max}}\) vs \(T\) curves of Fig. 5 have been analytically obtained by Eq. (8). Respectively, we obtain \(V_{\text{max}} = 3(1 - T^2/2LC_M)\) [the amplitude of Eq. (16)] and \(V_{\text{max}} = \sqrt{12(1 - T^2/2LC_M)}\). For any value of \(\varepsilon\), both curves are concave, while our model and the one obtained by numerical simulation of the propagating soliton give a convex dependence of \(V_{\text{max}}\) on \(T\). It seems that the nonlinear approximations (17) and (18) are not useful to describe the C–V characteristic of typical HBVs over a wide range of voltages. To give more insight on this, in Fig. 6 we have represented the three different C–V approximations considered in Fig. 5. According to Eq. (11), our approximation to the C–V curve of the HBV is a square pulse of width \(V_1\), height \(C_M\), and saturation capacitance \(C_0\). This approximation exactly describes the maximum and minimum values of the capacitance and provides a small error elsewhere. On the other hand, in spite of trying to properly choose the parameters \(C_M\) and \(\varepsilon\), expressions (17) and (18) are unable to adequately fit the C–V characteristic of a HBV in the wide range of voltages. In view of these comments and the results presented in this section, it is clear that previous approaches are not adequate to study soliton propagation in NLTLs loaded with HBVs. Alternatively, by means of a simple approximation to the C–V characteristic of HBVs, we have demonstrated that the wave forms of propagating solitons can be analytically obtained. The comparison to numerically simulated solitons in LC networks proves the validity of our approach.

VI. CONCLUSIONS

In conclusion we have reported a study on soliton propagation characteristics in periodically loaded NLTLs. A simple equation, valid for any nonlinear device, has been presented to directly determine the relationship between soliton amplitude and per-section propagation delay [Eq. (8)]. By means of a simple approximation, useful for moderate and high amplitude solitons, an analytical expression for the complete description of the solitaries has been presented [Eq. (13)]. The model is appropriate for symmetric C–V characteristics, typical of HBVs, as well as for large-signal analysis; both necessary conditions for high performance NLTL multipliers. The model has been compared to numerically simulated solitons in a nonlinear LC network. The results clearly justify the utility of our approach for the study of moderate to high voltage solitons in actual NLTLs. The presented model can be of help to understand soliton-like harmonic generation in NLTLs loaded with HBV for terahertz frequency multipliers.

ACKNOWLEDGMENTS

The authors would like to thank Lluis Torner and Juan Pérez for stimulating our interest in solitons and to the Quantum Devices and Terahertz Electronics Group of IEMN (Lille University) for helpful discussions about NLTL. This work has been supported by the DGES under Project No. PB97-0182.

9Strictly speaking, \(C_s\) includes the per-section capacitance of the line and the constant saturation capacitance of the nonlinear device.